# RELATIVITY AND COSMOLOGY I

**Solutions to Problem Set 11** Fall 2023

## 1. Penrose Diagrams

(a) The new coordinates $\bar{T}$ and $\bar{R}$ satisfy one of our criteria: they have finite ranges

$$-\frac{\pi}{2} < \bar{T} < \frac{\pi}{2}\,, \qquad 0 < \bar{R} < \frac{\pi}{2}\,. \tag{1}$$

The Minkowski metric in these coordinates is

$$ds^2 = -\frac{\mathrm{d}\bar{T}^2}{\cos^4\bar{T}} + \frac{\mathrm{d}\bar{R}^2}{\cos^4\bar{R}} + \tan^2\bar{R}\mathrm{d}\Omega^2\,. \tag{2}$$

Considering null radial geodesics, $ds^2 = 0$, acting with this tensor on two copies of the vector $\partial_{\bar{R}}$, we get

$$\frac{d\bar{T}}{d\bar{R}} = \pm\frac{\cos^2\bar{T}}{\cos^2\bar{R}} \neq \pm 1\,. \tag{3}$$

Light rays do not propagate at 45 degrees. We have to look a bit harder for suitable coordinates for our Penrose diagram.

(b) The metric in these coordinates is

$$ds^2 = -\frac{1}{2}(\mathrm{d}u\mathrm{d}v + \mathrm{d}v\mathrm{d}u) + \frac{1}{4}(v-u)^2\mathrm{d}\Omega^2\,, \tag{4}$$

and the coordinate ranges are

$$-\infty < u < \infty\,, \qquad -\infty < v < \infty \tag{5}$$

with the additional constraint that $u \leq v$ coming from their definitions in terms of $t$ and $r$.

(c) In Penrose coordinates, the metric becomes

$$ds^2 = \frac{1}{4\cos^2 U\cos^2 V}\left[-2(\mathrm{d}U\mathrm{d}V + \mathrm{d}V\mathrm{d}U) + \sin^2(V-U)\mathrm{d}\Omega^2\right] \tag{6}$$

and the coordinate ranges are

$$-\frac{\pi}{2} < U < \frac{\pi}{2}\,, \qquad -\frac{\pi}{2} < V < \frac{\pi}{2}\,, \tag{7}$$

with again the same additional constraint $U \leq V$.

(d) Now the final coordinate change. We get the metric

$$ds^2 = \frac{1}{(\cos T + \cos R)^2}\left[-\mathrm{d}T^2 + \mathrm{d}R^2 + \sin^2 R\mathrm{d}\Omega^2\right], \tag{8}$$

with coordinate ranges

$$0 \leq R < \pi\,, \qquad -\pi < T < \pi\,, \tag{9}$$

where we used that the constraint $U \leq V$ implies $R \equiv V - U$ is positive. Moreover, you can check that $T$ satisfies the following inequality

$$|T| < \pi - R\,. \tag{10}$$

The manifold in the square brackets is clearly $\mathbb{R} \times S^3$, also called the **Einstein static universe**. It was Einstein's initial proposal to describe our real cosmology, before knowing that our universe actually is expanding and is not static. You will see more about this in your next GR course. Because our coordinates only range over a finite interval of $T$, the statement we can make is that Minkowski space is **conformally related** to a finite portion of the Einstein static universe (see figure H.3 on Carroll for a visualization). Two spacetimes are said to be conformally related if there is some coordinate system in which

$$ds_1^2 = \omega(x)^2 ds_2^2\,, \tag{11}$$

where $\omega(x)$ is some function of the coordinates. The Penrose diagram we will draw is essentially built from this fictitious spacetime in the square brackets. Light rays in this spacetime propagate at 45 degrees

$$\frac{dR}{dT} = \pm 1\,. \tag{12}$$

(e) The ranges of the coordinates force us to draw this spacetime as a triangle. The full relations between $(T, R)$ and $(t, r)$ are

$$T = \arctan(t + r) + \arctan(t - r)\,, \qquad R = \arctan(t + r) - \arctan(t - r)\,. \tag{13}$$

The drawing is thus something like you see on Figure 1. On the figure, we have indicated the conventional names of some important regions on the Penrose diagram, namely we have

- Future timelike infinity $i^+$,
- Future lightlike infinity $\mathcal{I}^+$,
- Spacelike infiniy $i^0$,
- Past lightlike infinity $\mathcal{I}^-$,
- Past timelike infinity $i^-$.

It is important to mention that all future-directed timelike geodesics end up at $i^+$ while all null geodesics end up on $\mathcal{I}^+$

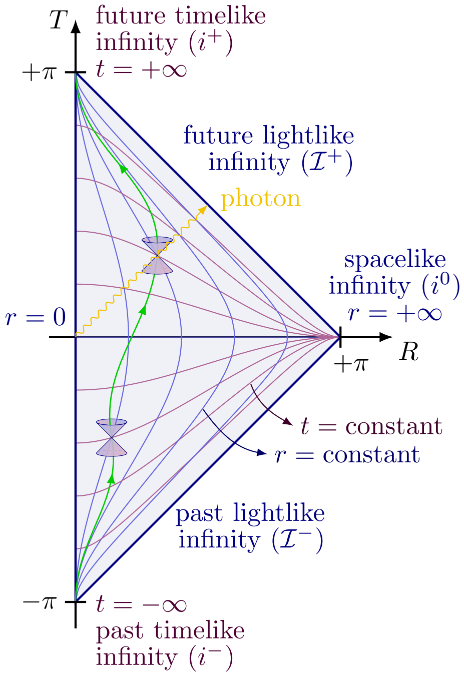


Figure 1: The Penrose diagram of Minkowski space. Credits to Izaak Neutelings.

## 2. A Hamiltonian Approach to Geodesics

(a) Let us start by computing the conjugate momenta

$$p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \frac{1}{\xi} g_{\mu\nu} \dot{x}^\nu \,. \tag{14}$$

We thus have

$$p_\mu g^{\mu\rho} = \frac{1}{\xi} g^{\mu\rho} g_{\mu\nu} \dot{x}^\nu \longrightarrow \dot{x}^\rho = \xi p^\rho \,. \tag{15}$$

The Hamiltonian is thus given by

$$\begin{aligned} \mathcal{H}(x, p, \xi) &= (p_\mu \dot{x}^\mu - \mathcal{L})\Big|_{\dot{x}^\mu = \xi p^\mu} \\ &= \xi \; p_\mu p^\mu - \frac{\xi}{2} p^\mu p_\mu + \frac{\xi}{2} m^2 \\ &= \frac{\xi}{2} \left[ p_\mu p^\mu + m^2 \right] . \end{aligned} \tag{16}$$

(b) The Hamilton equations are

$$\dot{x}^\mu = \frac{\partial \mathcal{H}}{\partial p_\mu} = \xi p^\mu \tag{17}$$

$$\dot{p}_\mu = -\frac{\partial \mathcal{H}}{\partial x^\mu} = -\frac{\xi}{2} p_\rho p_\nu \partial_\mu g^{\rho\nu} \,. \tag{18}$$

To find the geodesic equations we need to use (17) in (18), keeping in mind that derivatives with respect to $\lambda$ and raising and lowering indices are not operations that commute. Let us look more closely at the left hand side of (18):

$$\begin{aligned} \dot{p}_\mu &= \frac{1}{\xi}\frac{d}{d\lambda}\left(g_{\mu\nu}\dot{x}^\nu\right) \\ &= \frac{1}{\xi}\left(\dot{x}^\rho\partial_\rho g_{\mu\nu}\dot{x}^\nu + g_{\mu\nu}\ddot{x}^\nu\right) . \end{aligned} \tag{19}$$

The full equation is thus

$$\dot{x}^\rho\partial_\rho g_{\mu\nu}\dot{x}^\nu + g_{\mu\nu}\ddot{x}^\nu = \frac{1}{2}\dot{x}^\rho\dot{x}^\nu\partial_\mu g_{\rho\nu} , \tag{20}$$

where, to go from the right hand side of (18) to the right hand side of (20), we have used that[1]

$$\dot{x}_\rho\dot{x}_\nu\partial_\mu g^{\rho\nu} = -\dot{x}^\rho\dot{x}^\nu\partial_\mu g_{\rho\nu} . \tag{21}$$

Multiplying by $g^{\mu\lambda}$ and reshuffling terms, we get

$$\ddot{x}^\lambda + \frac{1}{2}g^{\mu\lambda}\left(\partial_\rho g_{\mu\nu} + \partial_\nu g_{\mu\rho} - \partial_\mu g_{\rho\nu}\right)\dot{x}^\rho\dot{x}^\nu = 0 , \tag{22}$$

where we recognize exactly the geodesic equations.

(c) The Hamilton equations of motion for $\xi$ are

$$\dot{\xi} = \frac{\partial\mathcal{H}}{\partial p_\xi} = 0 , \qquad \dot{p}_\xi = -\frac{\partial\mathcal{H}}{\partial\xi} = -\frac{1}{2}\left[p_\mu p^\mu + m^2\right] . \tag{23}$$

Since we computed that $p_\xi = 0$, necessarily we have that $\mathcal{H}_{\text{on-shell}} = 0$ . The constraint is thus $p_\mu p^\mu = -m^2$, which in terms of four velocities reads

$$-g_{\mu\nu}U^\mu U^\nu = \xi^2 m^2 \tag{24}$$

(d) From (18) we see that $p_\mu$ is a constant of motion if $\partial_\mu g^{\rho\nu} = 0$ .

(e) Let us expand the derivative of $f(x,p)$:

$$\frac{d}{d\lambda}f(x,p) = \frac{dx^\mu}{d\lambda}\frac{\partial f}{\partial x^\mu} + \frac{dp^\mu}{d\lambda}\frac{\partial f}{\partial p^\mu} . \tag{25}$$

Using the Hamilton equations of motion,

$$\begin{aligned} \frac{d}{d\lambda}f(x,p) &= \frac{\partial\mathcal{H}}{\partial p_\mu}\frac{\partial f}{\partial x^\mu} - \frac{\partial\mathcal{H}}{\partial x_\mu}\frac{\partial f}{\partial p^\mu} \\ &= \{f,\mathcal{H}\} , \end{aligned} \tag{26}$$

as we wanted to show.

(f) From what we just showed, if we choose $f(x,p) = p_\mu K^\mu(x)$ , we get

$$\begin{aligned} \dot{f} = \{f,\mathcal{H}\} = \frac{\partial f}{\partial x^\mu}\frac{\partial\mathcal{H}}{\partial p_\nu} - \frac{\partial f}{\partial p_\mu}\frac{\partial\mathcal{H}}{\partial x^\mu} &= (p_\alpha\partial_\mu K^\alpha)g^{\mu\beta}p_\beta - K^\mu\left(\frac{1}{2}\partial_\mu g^{\alpha\beta}p_\alpha p_\beta\right) = \\ &= p_\alpha p_\beta\left(g^{\mu\beta}\partial_\mu(g^{\alpha\gamma}K_\gamma) - \frac{1}{2}g^{\mu\gamma}K_\gamma\partial_\mu g^{\alpha\beta}\right) = \\ &= p_\alpha p_\beta\left(g^{\mu\beta}g^{\alpha\gamma}\partial_\mu K_\gamma + g^{\mu\beta}K_\gamma\partial_\mu g^{\gamma\alpha} - \frac{1}{2}g^{\mu\gamma}\partial_\mu g^{\alpha\beta}K_\gamma\right) = \\ &= p^\alpha p^\beta(\partial_\alpha K_\beta + g_{\sigma\alpha}\partial_\beta g^{\gamma\sigma}K_\gamma - \frac{1}{2}g_{\alpha\alpha'}g_{\beta\beta'}g^{\mu\gamma}\partial_\mu g^{\alpha'\beta'}K_\gamma) \end{aligned} \tag{27}$$

[1] Check that this is true by starting from $\partial_\mu\delta^\alpha_\beta = 0$.

Using identity $\partial_\xi g^{\alpha\beta} = -g^{\alpha\alpha'}g^{\beta\beta'}\partial_\xi g_{\alpha'\beta'}$ one arrives at

$$\dot{f} = p^\alpha p^\beta \left(\partial_\alpha K_\beta - g^{\mu\gamma}\partial_\beta g_{\mu\alpha}K_\gamma + \frac{1}{2}g^{\mu\gamma}\partial_\mu g_{\alpha\beta}K_\gamma\right) \tag{28}$$

By using the fact that the $\alpha$ and $\beta$ indices in the brackets are contracted with the symmetric quantity $p^\alpha p^\beta$, we can equivalently write

$$\begin{aligned}\dot{f} =& p^\alpha p^\beta \left(\partial_{(\alpha}K_{\beta)} - g^{\mu\gamma}\partial_{(\beta}g_{\alpha)\mu}K_\gamma + \frac{1}{2}g^{\mu\gamma}\partial_\mu g_{\alpha\beta}K_\gamma\right) = \\ =& p^\alpha p^\beta \left(\partial_{(\alpha}K_{\beta)} - \Gamma^\gamma_{(\alpha\beta)}K_\gamma\right) = p^\alpha p^\beta \nabla_{(\alpha}K_{\beta)}\end{aligned} \tag{29}$$

For general $p$ it vanishes iff $\nabla_{(\alpha}K_{\beta)} = 0$, which is Killing equation.

# 3. The vielbein formalism

(a) At each point $x$ the metric of a $d$ dimensional Lorentzian spacetime is a symmetric tensor that can be diagonalized as

$$P^\mu_a(x)g_{\mu\nu}(x)P^\nu_b(x) = \mathrm{diag}(\text{-}\lambda_0, \lambda_1, \ldots, \lambda_{\text{d-1}})_{ab}, \tag{30}$$

where the change of basis matrices $P^\mu_a(x)$ are functions of $x$ since the metric is in general a different matrix at each point. We can now define

$$e^\mu_a(x) = \sqrt{\lambda_a}P^\mu_a, \qquad \text{(no sum)} \tag{31}$$

such that

$$e^\mu_a g_{\mu\nu} e^\nu_b = \eta_{ab} \tag{32}$$

(b) Using the Lorentz matrices

$$\Lambda^a{}_c\Lambda^b{}_d\eta_{ab} = \eta_{cd}, \tag{33}$$

we get

$$(\Lambda^a{}_c e^\mu_a)g_{\mu\nu}(\Lambda^b{}_d e^\nu_b) = \eta_{cd}, \tag{34}$$

which coincides with (32) if we identify

$$e'^\mu_a = \Lambda^d{}_a e^\mu_d. \tag{35}$$

Therefore the frame fields are only defined up to Lorentz transformations.

(c) A direct computation yields

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = e^a_\mu e^b_\nu \eta_{ab}dx^\mu dx^\nu = e^a e^b \eta_{ab}. \tag{36}$$

(d) We have, first by relating the frame components $V^a$ to the vector components $V^\nu$

$$\begin{aligned}\nabla_\mu V^a &\equiv \partial_\mu V^a + \omega_\mu{}^a{}_b V^b \\ &= \partial_\mu(e^a_\nu V^\nu) + \omega_\mu{}^a{}_b V^b \\ &= e^a_\nu \partial_\mu V^\nu + (\partial_\mu e^a_\nu)V^\nu + \omega_\mu{}^a{}_b V^b\end{aligned} \tag{37}$$

On the other hand, we can think of $\nabla_\mu V^a$ as the components of a 2-tensor, with one frame component and one spacetime component. We can convert to full spacetime components via

$$\nabla_\mu V^a = e^a_\nu \nabla_\mu V^\nu = e^a_\nu \left(\partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho\right). \tag{38}$$

Comparing the two expressions for $\nabla_\mu V^a$ we get

$$e^a_\nu \Gamma^\nu_{\mu\rho} = \partial_\mu e^a_\rho + \omega_\mu{}^a{}_b e^b_\rho. \tag{39}$$

Multiplying by the inverse frame $e^\sigma_a$ this gives

$$\Gamma^\sigma_{\mu\rho} = e^\sigma_a \partial_\mu e^a_\rho + e^\sigma_a \omega_\mu{}^a{}_b e^b_\rho, \tag{40}$$

and multiplying instead by $e^\rho_c$ gives

$$e^\rho_c e^a_\nu \Gamma^\nu_{\mu\rho} - e^\rho_c \partial_\mu e^a_\rho = \omega_\mu{}^a{}_c. \tag{41}$$

(e) First we have

$$\nabla_\mu \eta_{ab} = -\omega_\mu{}^c{}_a \eta_{cb} - \omega_\mu{}^c{}_b \eta_{ac} = -\omega_{\mu ba} - \omega_{\mu ab} = 0, \tag{42}$$

which implies $\omega_{\mu[ab]} = 0$. Then we have

$$de^a = \partial_\nu e^a_\mu dx^\nu \wedge dx^\mu, \tag{43}$$

such that

$$\begin{aligned} de^a + \omega^a{}_b \wedge e^b &= \partial_\mu e^a_\nu dx^\mu \wedge dx^\nu + \omega_\mu{}^a{}_b e^b_\nu dx^\mu \wedge dx^\nu \\ &= \left(\partial_\mu e^a_\nu + \omega_\mu{}^a{}_b e^b_\nu\right) dx^\mu \wedge dx^\nu \end{aligned} \tag{44}$$

The term inside the parentheses is exactly (39), hence

$$de^a + \omega^a{}_b \wedge e^b = e^a_\lambda \Gamma^\lambda_{\nu\mu} dx^\mu \wedge dx^\nu = 0 \tag{45}$$

where for the last equality we assumed the symmetric Levi-Civita connection. More generally there could be a torsion term. For the Riemann tensor we start from the defining equation

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho_{\lambda\mu\nu} V^\lambda = e^\rho_a (R^a{}_b)_{\mu\nu} V^b \tag{46}$$

We compute

$$\nabla_\mu \nabla_\nu V^a = \partial_\mu \nabla_\nu V^a - \Gamma^\lambda_{\mu\nu} \nabla_\lambda V^a + \omega_\mu{}^a{}_b \nabla_\nu V^b \tag{47}$$

and then we multiply by $dx^\mu \wedge dx^\nu$ which will antisymmetrize over $\mu$ and $\nu$. In particular the second term will vanish. We are left with

$$\nabla_\mu \nabla_\nu V^a = \partial_\mu \omega_\nu{}^a{}_b V^b + \omega_\nu{}^a{}_b \partial_\mu V^b + \omega_\mu{}^a{}_b \omega_\nu{}^b{}_c V^c + \omega_\mu{}^a{}_b \partial_\nu V^b + \ldots \tag{48}$$

where ... contains terms symmetric in $\mu, \nu$. Note that the second term and the last term form a combination that is also symmetric in $\mu\nu$. Hence we have

$$\nabla_\mu \nabla_\nu V^a = \left(\partial_\mu \omega_\nu{}^a{}_b + \omega_\mu{}^a{}_c \omega_\nu{}^c{}_b\right) V^b + \ldots \tag{49}$$

from which we can read off

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b. \tag{50}$$

(f) From the metric

$$ds^2 = -dt^2 + a^2(t)dx^i dx^i = e^a e^b \eta_{ab} \tag{51}$$

we can directly read off the 1-form frame fields

$$e^0 = dt, \qquad e^i = a(t)dx^i. \tag{52}$$

The $t$ and $i$ components of the first structure equation read

$$\begin{aligned} 0 &= de^0 + \omega^0_a \wedge e^a = a(t)\omega^0{}_j \wedge dx^j, \\ 0 &= de^i + \omega^i_a \wedge e^a = \dot{a}(t)dt \wedge dx^i + \omega^i{}_0 \wedge dt + a(t)\omega^i_j \wedge dx^j. \end{aligned} \tag{53}$$

Not that from the antisymmetry of $\omega_{ab}$ we get $\omega^0{}_i = \omega^i{}_0$. The first equation implies $\omega^0{}_j = f(t)dx^j$ for some $f(t)$. Plugging in the second equation we get

$$-\dot{a}dx^i \wedge dt + f(t)dx^i \wedge dt + a\omega^i{}_j dx^j = 0, \tag{54}$$

which is solved by

$$f(t) = \dot{a}, \qquad \omega^0{}_j = \dot{a}dx^j, \qquad \omega^i{}_j = 0. \tag{55}$$

We now write the components of the second structure equation

$$\begin{aligned} R^0{}_i &= d\omega^0{}_i + \omega^0{}_c \omega^c{}_i = \ddot{a}dt \wedge dx^i, \\ R^i{}_j &= d\omega^i{}_j + \omega^i{}_c \omega^c{}_j = \dot{a}^2 dx^i \wedge dx^j. \end{aligned} \tag{56}$$

We can read the Riemann tensor from

$$R^\lambda{}_{\rho\mu\nu} = e^\lambda_a e^b_\rho \left(R^a{}_b\right)_{\mu\nu}, \tag{57}$$

which gives the non zero components

$$R^0{}_{i0j} = a\ddot{a}\delta_{ij}, \qquad R^i{}_{jkl} = \dot{a}^2 \left(\delta^i_k \delta_{jl} - \delta^i_l \delta_{jk}\right). \tag{58}$$

The Ricci is obtained by taking traces

$$R_{00} = -3\frac{\ddot{a}}{a}, \qquad R_{ij} = (a\ddot{a} + 2\dot{a}^2)\delta_{ij}. \tag{59}$$

and the Einstein equations are $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}$, where we find

$$R = 6\frac{\ddot{a}}{a} + 6\frac{\dot{a}^2}{a^2}. \tag{60}$$

For a perfect fluid $T_{\mu\nu} = \mathrm{diag}(\rho, p, p, p)$, and we get

$$\frac{\dot{a}^2}{a} = \frac{8}{3}\pi G\rho, \qquad \frac{\ddot{a}}{a} = -\frac{4}{3}\pi G(\rho + 3p). \tag{61}$$

Those are known as the Friedmann equations and are of extreme importance for cosmology.